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FAST TRACK COMMUNICATION

# Delay induces quasi-periodic vibrational resonance

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**Abstract**
In this communication, the delayed bistable system under the excitation of two different periodic signals is investigated by numerical simulation. It is found that the delay can induce quasi-periodic vibrational resonance. With the cooperation of the strong high-frequency signal and the delay, the weak low-frequency signal in the delayed bistable system can be enhanced greatly.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Recently, a phenomenon that is called vibrational resonance (VR) is investigated by numerical or analytical treatments [1, 2]. VR is commonly said to occur when a nonlinear system subjected to two different periodic signals and the weak low-frequency periodic signal can be amplified by increasing the amplitude of the strong high-frequency signal. VR is similar to the famous stochastic resonance (SR) in that the high-frequency signal is replaced by the noise [3]. It should be mentioned that the system modulated by two different frequency signals is interested in commutation technologies [4], acoustics [5], neuroscience [6] or laser physics [7]. For these reasons, VR has attracted more and more attention [8–15].

In real systems, time is needed to transmit the information, energy, etc. As a result, the delay should be introduced in the dynamical system. To our knowledge, delay has not been considered in VR till now. So in this communication, we investigate the effects of delay on VR in a delayed bistable system.

We consider the delayed bistable system under the excitation of two different periodic signals, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -x^3(t) + x(t-\alpha) + f\cos\omega t + F\cos\Omega t. \tag{1}$$

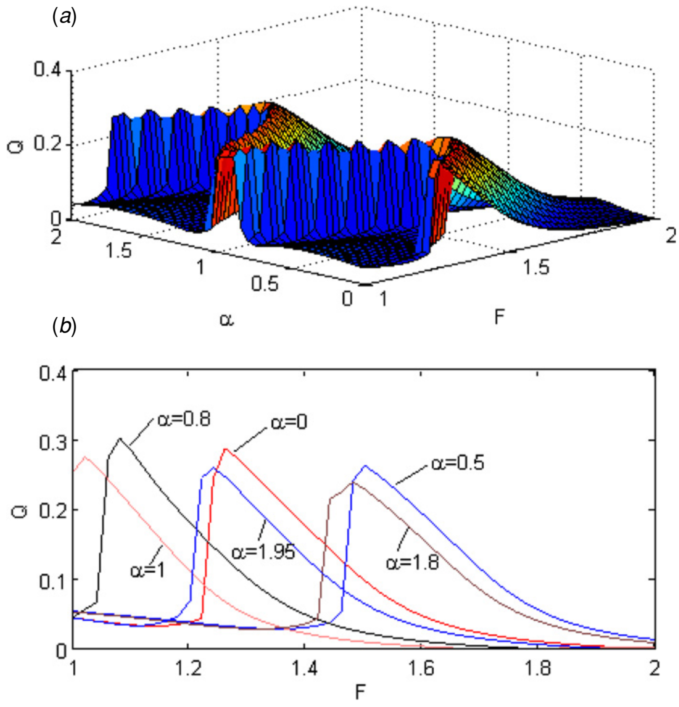


**Figure 1.** Dependence of the response amplitude $Q$ on $F$ and $\alpha$ for $f = 0.02$, $\omega = 0.01$, $\Omega = 5$.

In equation (1), $\alpha \geqslant 0$ is the time delay and $f \ll F$, $\omega \ll \Omega$. To quantify the occurrence of VR, we use the response amplitude $Q$ of the system at the lower frequency $\omega$, which is given by $Q = \sqrt{B_s^2 + B_c^2}$ with

$$B_s = \frac{2}{nT}\int_0^{nT} x(t)\sin\omega t\,\mathrm{d}t, \qquad B_c = \frac{2}{nT}\int_0^{nT} x(t)\cos\omega t\,\mathrm{d}t, \tag{2}$$

where $T = 2\pi/\omega$ and $n$ is an positive integer.

## 2. Quasi-periodic vibrational resonance and the optimal high frequency

In this section, we calculate the response amplitude $Q$ numerically with the fourth-order Runge–Kutta method with fixed step sizes $\Delta t = 0.01$, $\Delta F = 0.01$ and $\Delta\alpha = 0.01$. The initial condition is $x(0) = -1$, and the total time is $t = 4000$. The effects of the delay on the response of the system to the low-frequency signal are investigated according to the results of $Q$.

The curved surface of the response amplitude $Q$ with respect to the delay $\alpha$ and the signal amplitude $F$ is depicted in figure 1($a$). In view of this figure, one can see that the peak appears with the increase of the delay $\alpha$ or the signal amplitude $F$, and the maximum of the curve varies with $\alpha$ and $F$. In figure 1($b$), the evolution of the response amplitude $Q$ versus the signal amplitude $F$ is shown clearly. As $F$ increases, the response amplitude $Q$ increases and reaches the maximal value but decreases with the further increase in $F$. This phenomenon is the conventional VR since the occurrence is due to the high-frequency signal. Figure 2($a$) gives the evolution of the maximum of the response amplitude $Q$ with the increase of the delay. It is interesting and inconceivable that the maximal value of the response amplitude $Q_{\max}$ is a

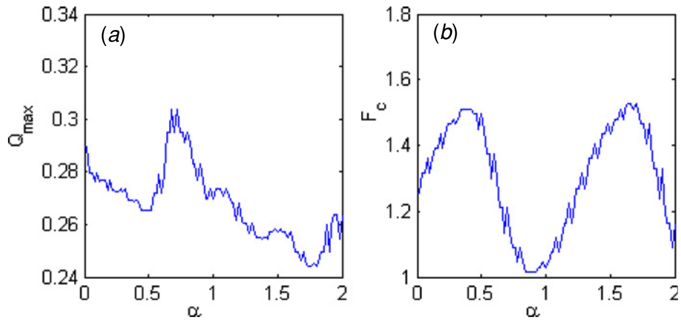


**Figure 2.** (*a*) The maximal response amplitude $Q_{\max}$ versus the delay. (*b*) The critical signal amplitude $F_c$ versus the delay for $f = 0.02$, $\omega = 0.01$, $\Omega = 5$.

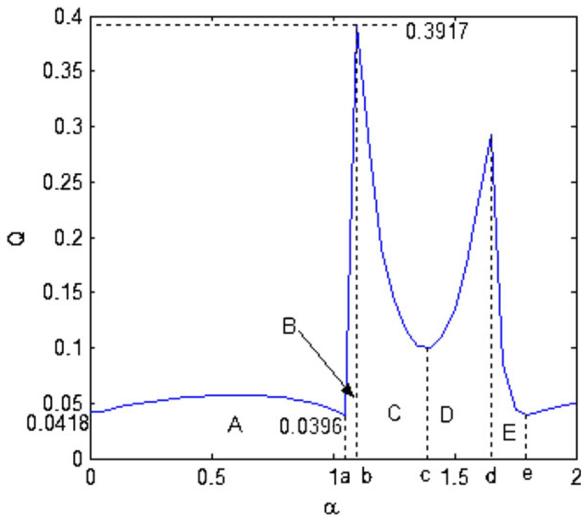


**Figure 3.** Dependence of the response amplitude $Q$ on $\alpha$ for $f = 0.02$, $\omega = 0.01$, $\Omega = 3.5$, $F = 1.0$.

nonlinear function of the delay, and the curve is similar to a random plot. This is because, for long time behavior, a very small variation in the delay can cause a large variation in the output (see figures 3 and 4). In figure 2(*b*), $F_c$ is the critical amplitude of the high-frequency signal that makes the response amplitude $Q$ reach the maximum, and it is a quasi-periodic function of the delay. From figures 1 and 2, we can see that the response amplitude $Q$ is influenced by the delay.

Figure 3 gives the response amplitude $Q$ versus the delay $\alpha$ for invariable signals. With the increase of the delay $\alpha$, there are two obvious peaks in the curve, and the curve in this figure essentially contains five regions, i.e. region A: $0 \leqslant \alpha \leqslant$ a (=1.05); region B: a $< \alpha \leqslant$ b (=1.1); region C: b $< \alpha \leqslant$ c (=1.4); region D: c $< \alpha \leqslant$ d (=1.65); region E: d $< \alpha \leqslant$ e (=1.8). In region A, the response amplitude $Q$ varies smoothly and slowly with the delay $\alpha$. What is noteworthy is region B, in which the response amplitude $Q$ amplified greatly when the delay increases in a small range. It goes up to 0.3917 from 0.0396 when the delay increases from 1.05 to 1.1, and this suggests that the response to the weak low-frequency signal is hypersensitive to the delay in region B. In addition, comparing with the delay-free

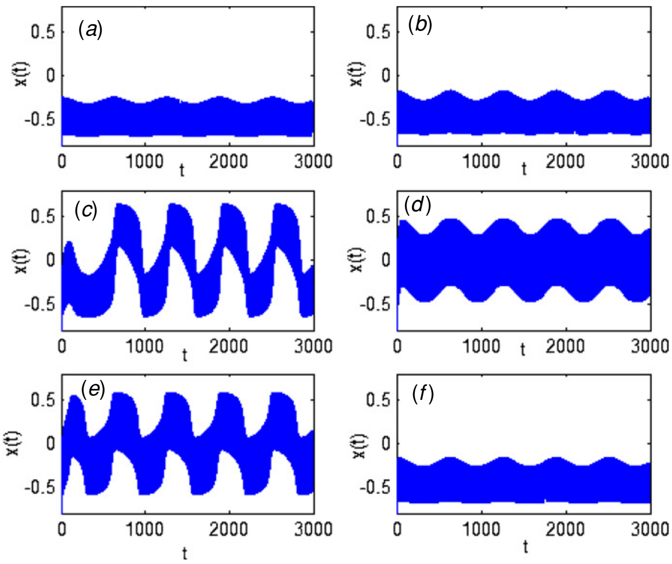


**Figure 4.** The output of the system with parameters $f = 0.02$, $\omega = 0.01$, $F = 1.0$, $\Omega = 3.5$: (*a*) $\alpha = 0$, (*b*) $\alpha = 1.05$, (*c*) $\alpha = 1.1$, (*d*) $\alpha = 1.4$, (*e*) $\alpha = 1.65$, (*f*) $\alpha = 1.8$.

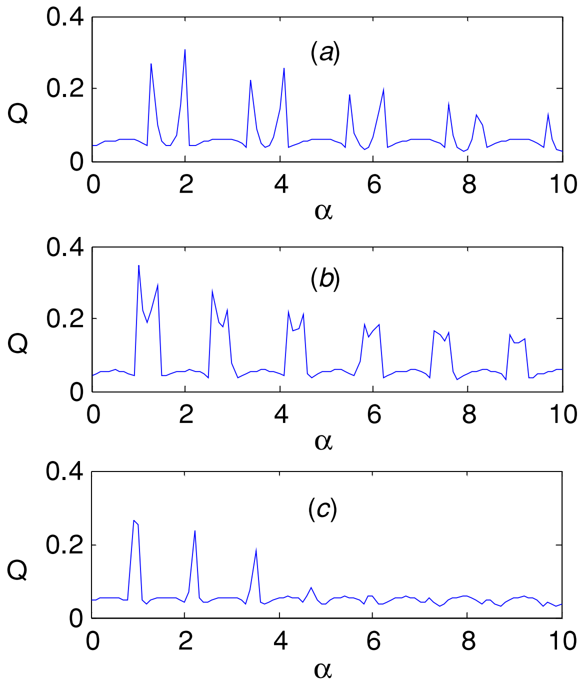


**Figure 5.** Dependence of the response amplitude $Q$ on $\alpha$ for $f = 0.02$, $F = 1.0$, $\omega = 0.01$, (*a*) $\Omega = 3$, (*b*) $\Omega = 4$, (*c*) $\Omega = 5$.

system (i.e. $\alpha = 0$), the response amplitude $Q$ at $\alpha = 1.1$ is improved greatly. In region D, the response amplitude $Q$ increases with the increase of the delay too. The difference between

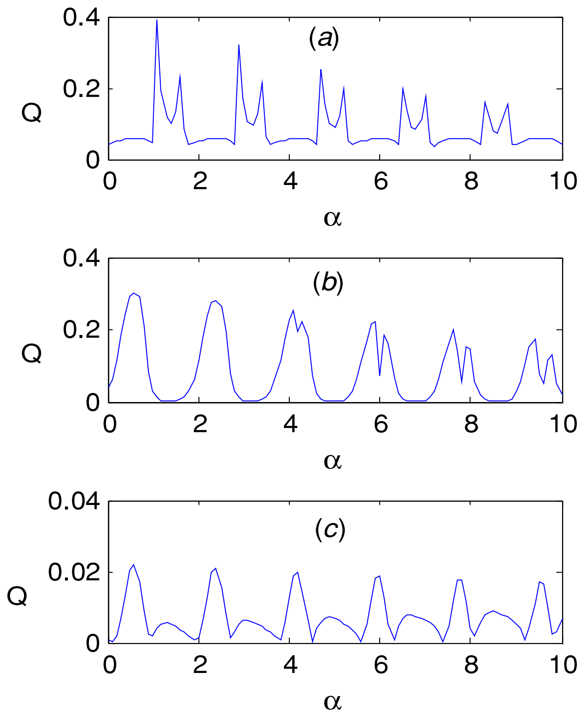


**Figure 6.** Dependence of the response amplitude $Q$ on $\alpha$ for $f = 0.02$, $\omega = 0.01$, $\Omega = 3.5$: (*a*) $F = 1.0$, (*b*) $F = 1.5$, (*c*) $F = 2.0$.

regions B and D is the sensitivity of the response amplitude $Q$ to the delay $\alpha$. In regions C and E, the response amplitude $Q$ is a decrease function of the delay. At $\alpha = 1.8 \approx 2\pi/\Omega$, the response amplitude $Q$ is approximated equivalent to that of the delay-free system. From this figure, we can see that the delay can improve the response amplitude $Q$. In other words, with the cooperation of the delay and the high-frequency signal, the weak low-frequency signal in a nonlinear system can be enhanced greatly. It is very important in the engineering fields or nature science.

Figure 4 shows the trajectory plots for the labeled delays in figure 3. For $0 \leqslant \alpha \leqslant$ a, that is in region A, the obits are confined to one well only. There is no cross-well motion (cf figures 4(*a*), (*b*) for $\alpha = 0$ and 1.05). At $\alpha = 1.1$ that makes the response amplitude $Q$ reaches the maximum in figure 3, the orbit lies in one well during one half of the drive cycle of the low-frequency signal and in the other well during the residual half of the cycle as is shown in figure 4(*c*). It is the celebrated input–output synchronization phenomenon that the particle transmits between the two wells regularly with the period of the weak low-frequency signal. This is the principle that the weak low-frequency signal amplified in VR. In figures 4(*d*) and (*e*), the synchronization phenomenon also presents, but they are weaker than that in figure 4(*c*). At $\alpha = 1.8$, as is shown in figure 4(*f*), the motion of the orbit is confined to one well again.

Figures 5 and 6 give the response amplitude $Q$ versus the delay $\alpha$ for different frequencies or amplitudes of the high-frequency signal. In these two figures, the delay varies in a larger range than that in figure 3. They can show the effects of a larger delay on the response amplitude $Q$. In the curves, with the increase of the delay, the peaks present in turn. In

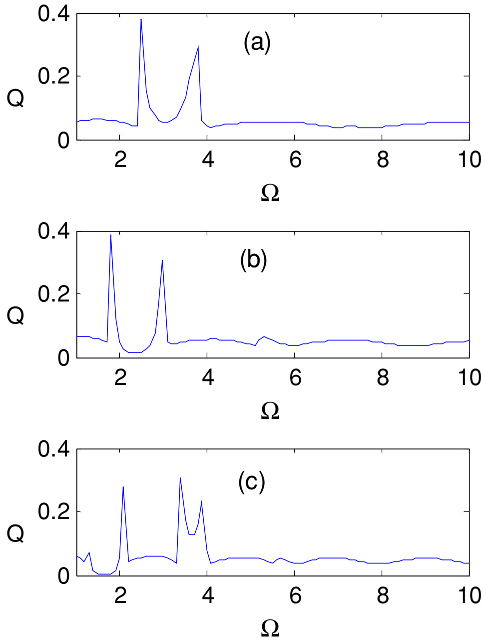


**Figure 7.** Dependence of the response amplitude $Q$ on $\Omega$ for $f = 0.02$, $\omega = 0.01$, $F = 1.0$: $(a)$ $\alpha = 1.5$, $(b)$ $\alpha = 2$, $(c)$ $\alpha = 3$.

addition, it can be found that the appearance of the peak is approximated equal to the cycle of the high-frequency signal. So in this communication, we call this phenomenon that is induced by the delay as quasi-periodic vibrational resonance. With the increase of the frequency or the amplitude of the high-frequency signal, the curve in one period turns from double peaks to a single peak as is shown in figures 5 and 6 from $(a)$–$(c)$. It is similar to the P-bifurcation that is a bifurcation behavior in the random dynamics. In a delay-free system, the resonant condition is determined by the signals and the parameters of the system [2, 12]. However, in a delayed system, the resonant condition should have much to do with the delay besides the parameters in the related delay-free system. In figures 5$(c)$ and 6$(c)$, the number of the delays that satisfy the resonant condition in one period decreases from 2 to 1 with the increase of the signal frequency $\Omega$ or the signal amplitude $F$. As a result, the double-peak maxima join together.

Figure 7 presents the curves of the response amplitude $Q$ versus the high frequency $\Omega$ with different delays. For the bistable system with fixed delay feedback, figure 7 shows that there are only a few $\Omega$ that make the response amplitude $Q$ achievable at the maxima. The high frequency cannot induce the quasi-periodic vibrational resonance phenomenon. It indicates that there are only a few high frequencies which can satisfy the resonant condition in the bistable system with a fixed delay feedback. We call the signal frequency $\Omega$ that makes the response amplitude $Q$ reaches its maximum as the optimal high frequency.

## 3. Conclusion

In this communication, we investigate the delayed bistable system under the excitation of two periodic signals with different frequencies. It is found that the delay can induce the quasi-periodic vibrational resonance. The output of the system presents the input–output synchronization phenomenon when the response amplitude to the weak signal is achieved at the maximum. Via the cooperation of the high-frequency signal and the delay, the weak low-frequency signal can be enhanced greatly. This supplies a new way for the amplification or recovery of the weak low-frequency signal in communication technologies, acoustics, neuroscience, engineering fields, etc. And in addition, it gives advice on system designs.

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